

An exponential estimate for Hilbert space-valued Ornstein–Uhlenbeck processes

Lukas Wresch

Faculty of Mathematics, Bielefeld University, Germany, E-mail: wresch@math.uni-bielefeld.de

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Abstract

Let Z be a H -valued Ornstein–Uhlenbeck process, $b: [0, 1] \times H \rightarrow H$ and $h: [0, 1] \rightarrow H$ be a bounded, Borel measurable functions with $\|b\|_\infty \leq 1$ then

$\mathbb{E} \exp \alpha \left| \int_0^1 b(t, Z_t + h(t)) - b(t, Z_t) dt \right|_H^2 \leq C$ holds, where the constant C is an absolute constant and $\alpha > 0$ depends only on the eigenvalues of the drift term of Z and $\|h\|_\infty$, the norm of h , in an explicit way. Using this we furthermore prove a concentration of measure result and estimate the moments of the above integral.

1 Introduction

Let H be a separable Hilbert space over \mathbb{R} with an orthonormal basis $(e_n)_{n \in \mathbb{N}}$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty[}, \mathbb{P})$ be a filtered stochastic basis with sigma-algebra \mathcal{F} , a right-continuous, normal filtration $\mathcal{F}_t \subseteq \mathcal{F}$ and a probability measure \mathbb{P} such that there is a cylindrical \mathcal{F}_t -Brownian motion $(B_t)_{t \in [0, \infty[}$ taking values in $\mathbb{R}^\mathbb{N}$ which is $\mathcal{F}/\mathcal{B}(\mathcal{C}([0, \infty[, \mathbb{R}^\mathbb{N}))$ measurable, where $\mathcal{B}(\mathcal{C}([0, \infty[, \mathbb{R}^\mathbb{N}))$ denotes the Borel sigma-algebra. We define the Wiener measure $\mathcal{W} := B(\mathbb{P}) := \mathbb{P}[B^{-1}]$ as the image measure of B under \mathbb{P} . Let $A: D(A) \rightarrow H$ be a positive definite, self-adjoint, closed, densely defined operator such that A^{-1} is trace-class and

$$Ae_n = \lambda_n e_n, \quad \lambda_n > 0, \quad \forall n \in \mathbb{N}.$$

This implies that

$$\sum_{n \in \mathbb{N}} \lambda_n^{-1} =: \Lambda < \infty. \quad (1.0.1)$$

By fixing the basis $(e_n)_{n \in \mathbb{N}}$ we identify H with ℓ^2 , so that $H \cong \ell^2 \subseteq \mathbb{R}^\mathbb{N}$. Let $(Z_t^A)_{t \in [0, \infty[}$ be an H -valued (actually ℓ^2 -valued) Ornstein–Uhlenbeck process which has continuous sample paths

with so-called drift term A , i.e. a strong solution to

$$dZ_t^A = -AZ_t^A dt + dB_t$$

with initial condition $Z_0^A = 0$. Furthermore, we define the Ornstein–Uhlenbeck measure \mathbb{P}_A as

$$\mathbb{P}_A[F] := \mathbb{P} \left[(Z^A)^{-1} (F) \right], \quad \forall F \in \mathcal{B}(\mathcal{C}([0, \infty[, \ell^2)).$$

In this article we show that there exist a absolute constant C and an $\alpha_A > 0$ such that

$$\mathbb{E} \exp \left(\frac{\alpha_A}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^A + h(t)) - b(t, Z_t^A) dt \right|_H^2 \right) \leq C \quad (1.0.2)$$

holds, where α_A only depends on the eigenvalues of the drift term A of Z^A . By approximating b with smooth functions and using the Fundamental Theorem of Calculus it suffices to prove that

$$\mathbb{E} \exp \left(\alpha_A \left| \int_0^1 b'(t, Z_t^A) dt \right|_H^2 \right) \leq C \quad (1.0.3)$$

holds for one-dimensional Ornstein–Uhlenbeck processes. In order to prove (1.0.3) we follow [Sha14] and split

$$\int_0^1 b'(t, Z_t^A) dt = \int_0^1 b(t, Z_t^A) d^*t - \int_0^1 b(t, Z_t^A) dt$$

into a forward and backward integral. Using explicit knowledge of the time-reversed process \overleftarrow{Z}^A the estimate (1.0.3) is proved in Proposition 2.1 and extended in Lemma 2.2 to infinite dimensions. The main result is then proved in Theorem 2.3. Furthermore, we show in Corollaries 3.1 and 3.2 two applications of this result.

2 Exponential estimate

Proposition 2.1

There exists an absolute constant $C \in \mathbb{R}$ and a non-increasing map

$$\alpha :]0, \infty[\longrightarrow]0, \infty[$$

$$\lambda \longmapsto \alpha_\lambda$$

with

$$\alpha_\lambda e^{2\lambda} \lambda^{-1} \geq \frac{e}{1152}, \quad \forall \lambda > 0.$$

such that for all one-dimensional Ornstein–Uhlenbeck processes $(Z_t^\lambda)_{t \in [0, \infty[}$ with drift term $\lambda > 0$, i.e.

$$\begin{cases} dZ_t^\lambda = -\lambda Z_t^\lambda dt + dB_t, \\ Z_0^\lambda = 0. \end{cases}$$

and for all Borel measurable functions $b: [0, 1] \times \mathbb{R} \longrightarrow H$, which are in the second component twice continuously differentiable with

$$\|b\|_\infty := \sup_{t \in [0, 1], x \in \mathbb{R}} |b(t, x)|_H \leq 1.$$

The following inequality

$$\mathbb{E} \exp \left(\alpha_\lambda \left| \int_0^1 b'(t, Z_t^\lambda) dt \right|_H^2 \right) \leq C \leq 3$$

holds, where b' denotes the first derivative of b w.r.t. the second component x .

Proof

Let $(Z_t^\lambda)_{t \in [0, \infty[}$ be a one-dimensional Ornstein–Uhlenbeck process, i.e. a strong solution to

$$dZ_t^\lambda = -\lambda Z_t^\lambda dt + dB_t,$$

where $\lambda > 0$, $Z_0^\lambda = 0$ and let $b: [0, 1] \times \mathbb{R} \longrightarrow H$ be as in the assertion. Define

$$Y_s := b(s, Z_s^\lambda), \quad \forall s \in [0, 1]$$

and denote by $(Y^n)_{n \in \mathbb{N}}$ the components of Y . Then by [BJ97, Remark 2.5] we have for every $n \in \mathbb{N}$

$$\langle Y^n, Z^\lambda \rangle_1 = \int_0^1 b'_n(s, Z_s^\lambda) d\langle Z^\lambda \rangle_s = \int_0^1 b'_n(s, Z_s^\lambda) ds,$$

where b_n is the n -th component of b and the quadratic covariation $\langle Y^n, Z^\lambda \rangle_t$ is the uniform in probability limit of

$$\sum_{\substack{t_i, t_{i+1} \in \mathbb{D}_m \\ 0 \leq t_i \leq t}} \left[Y_{t_{i+1}}^n - Y_{t_i}^n \right] \cdot \left[Z_{t_{i+1}}^\lambda - Z_{t_i}^\lambda \right].$$

Moreover, applying [BJ97, Corollary 2.3] results in

$$\int_0^1 b'_n(s, Z_s^\lambda) ds = \langle Y^n, Z^\lambda \rangle_1 = \int_0^1 Y_s^n d^* Z_s^\lambda - \int_0^1 Y_s^n dZ_s^\lambda, \quad (2.1.1)$$

where the backward integral is defined as

$$\int_0^t Y_s^n d^* Z_s^\lambda := - \int_{1-t}^1 Y_s^n d^{\leftarrow} Z_s^\lambda, \quad \forall t \in [0, 1] \quad (2.1.2)$$

and

$$\overset{\leftarrow}{X}_s := X_{1-s}, \quad \forall s \in [0, 1]$$

denotes the time-reversal of a generic stochastic process X . Since (2.1.1) holds for all components $n \in \mathbb{N}$ we also have

$$\int_0^1 b'(s, Z_s^\lambda) ds = \langle Y, Z^\lambda \rangle_1 = \int_0^1 Y_s d^* Z_s^\lambda - \int_0^1 Y_s dZ_s^\lambda, \quad (2.1.3)$$

where $\langle Y, Z^\lambda \rangle$ is defined as $(\langle Y^n, Z^\lambda \rangle)_{n \in \mathbb{N}}$.

Z^λ is an Itô diffusion process with generator

$$L_t = a(t, x) \nabla_x + \frac{1}{2} \sigma(t, x) \Delta_x = -\lambda x \nabla_x + \frac{1}{2} \Delta_x.$$

I.e. $a(t, x) = -\lambda x$ and $\sigma(t, x) = 1$. The probability density of Z_t^λ w.r.t. Lebesgue measure is

$$p_t(x) = \sqrt{\frac{\lambda}{\pi(1 - e^{-2\lambda t})}} e^{-\lambda x^2 / (1 - e^{-2\lambda t})}.$$

Observe that a and σ fulfill the conditions of [MNS89, Theorem 2.3], hence, the drift term $\overset{\leftarrow}{a}$ and diffusion term $\overset{\leftarrow}{\sigma}$ of the generator $\overset{\leftarrow}{L}_t$ of the time-reversed process $\overset{\leftarrow}{Z}^\lambda$, is given by

$$\overset{\leftarrow}{a}(t, x) = -a(1-t, x) + \frac{1}{p_{1-t}(x)} \nabla_x (\sigma(1-t, x) p_{1-t}(x)) = \left(\lambda - \frac{2\lambda}{1 - e^{2\lambda(1-t)}} \right) x$$

and

$$\overset{\leftarrow}{\sigma}(t, x) = \sigma(1-t, x) = 1.$$

Therefore (see [BR07, Remark 2.4]), we obtain

$$\overleftarrow{Z}_t^\lambda = \overleftarrow{Z}_0^\lambda + \overleftarrow{W}_t + \int_0^t \overleftarrow{Z}_s^\lambda \left(\lambda - \frac{2\lambda}{1 - e^{2\lambda(s-1)}} \right) ds, \quad (2.1.4)$$

where \overleftarrow{W}_t is a new Brownian motion defined by this equation. Set

$$\mathcal{G}_t^0 := \sigma \left(\overleftarrow{W}_s - \overleftarrow{W}_t, t \leq s \leq 1 \right)$$

and let $\tilde{\mathcal{G}}_t$ be the completion of \mathcal{G}_t^0 . Define

$$\mathcal{G}_t := \sigma \left(\tilde{\mathcal{G}}_{1-t} \cup \sigma(Z_1^\lambda) \right)$$

then \overleftarrow{W}_t is a \mathcal{G}_t -Brownian motion (see [Par86]). In conclusion we have by combining Equation (2.1.3) with (2.1.2)

$$- \int_0^1 b'(s, Z_s^\lambda) ds = \int_0^1 b(1-s, \overleftarrow{Z}_s^\lambda) d\overleftarrow{Z}_s^\lambda + \int_0^1 b(s, Z_s^\lambda) dZ_s^\lambda.$$

By plugging in (2.1.4) this is equal to

$$\underbrace{\int_0^1 b(1-s, \overleftarrow{Z}_s^\lambda) d\overleftarrow{W}_s}_{=: I_1} + \underbrace{\int_0^1 b(1-s, \overleftarrow{Z}_s^\lambda) \overleftarrow{Z}_s^\lambda \left(\lambda - \frac{2\lambda}{1 - e^{2\lambda(s-1)}} \right) ds}_{=: I_2} + \underbrace{\int_0^1 b(s, Z_s^\lambda) dZ_s^\lambda}_{=: I_3} \\ = I_1 + I_2 + I_3 =: I.$$

Observe that by (2.1.4) and the Yamada–Watanabe Theorem (see [RSZ08, Theorem 2.1]) $\overleftarrow{Z}_t^\lambda$ is a strong solution of an SDE driven by the noise \overleftarrow{W}_t , hence, $\overleftarrow{Z}_t^\lambda$ is \mathcal{G}_t -measurable so that the stochastic integral I_1 makes sense. In conclusion we get

$$\mathbb{E} \exp \left(\alpha_\lambda \left| \int_0^1 b'(t, Z_t^\lambda) dt \right|_H^2 \right) = \mathbb{E} \exp(\alpha_\lambda |I|_H^2) = \mathbb{E} \exp(\alpha_\lambda |I_1 + I_2 + I_3|_H^2), \quad (2.1.5)$$

for α_λ to be defined later. We will estimate the terms I_1 , I_2 and I_3 separately.

Estimate for I_1 :

Define

$$M_t := \int_0^t b(1-s, \overleftarrow{Z}_s^\lambda) d\overleftarrow{W}_s, \quad \forall t \in [0, 1].$$

Observe that $(M_t)_{t \in [0,1]}$ is a $(\mathcal{G}_t)_{t \in [0,1]}$ -martingale with $M_0 = 0$. Also note the following estimate for the quadratic variation of M

$$0 \leq \langle M \rangle_t \leq \int_0^t \|b\|_\infty^2 ds \leq \|b\|_\infty^2 \leq 1, \quad \forall t \in [0, 1].$$

In the next step we use the Burkholder–Davis–Gundy Inequality for time-continuous martingales with the optimal constant. In the celebrated paper [Dav76, Section 3] it is shown that the optimal constant in our case is the largest positive root of the Hermite polynomial of order $2k$. We refer to the appendix of [Ose12] for a discussion of the asymptotic of the largest positive root. See also [Kho14, Appendix B], where a self-contained proof of the Burkholder–Davis–Gundy inequality with asymptotically optimal constant can be found for the one-dimensional case. A proof for H -valued martingales can be obtained by a slight modification of [Kho14, Theorem B.1] to \mathbb{R}^d -valued martingales and by projecting H onto \mathbb{R}^d . The optimal constants in different cases is discussed in the introduction of [Wan91]. We have

$$\mathbb{E}|I_1|_H^{2k} = \mathbb{E}|M_1|_H^{2k} \leq 2^{2k} (2k)^k \underbrace{\mathbb{E}|\langle M \rangle_1|_H^k}_{\leq 1} \leq 2^{3k} \underbrace{k^k}_{\leq 2^{2k} k!} \leq 2^{5k} k!.$$

Choosing $\alpha_1 = \frac{1}{64}$ we obtain

$$\mathbb{E} \exp(\alpha_1 |I_1|_H^2) = \mathbb{E} \sum_{k=0}^{\infty} \frac{\alpha_1^k |I_1|_H^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{\alpha_1^k \mathbb{E}|I_1|_H^{2k}}{k!} \leq \sum_{k=0}^{\infty} 2^{-k} = 2 =: C_1.$$

Estimate for I_2 :

We have for any $\alpha_2^{(\lambda)} > 0$ to be specified later

$$\begin{aligned} \mathbb{E} \exp \alpha_2^{(\lambda)} |I_2|_H^2 &= \mathbb{E} \exp \alpha_2^{(\lambda)} \left| \int_0^1 b(1-t, \overleftarrow{Z}_t^\lambda) \overleftarrow{Z}_t^\lambda \lambda \left(1 - \frac{2}{1 - e^{2\lambda(t-1)}} \right) dt \right|_H^2 \\ &\leq \mathbb{E} \exp \alpha_2^{(\lambda)} \left| \int_0^1 \underbrace{|b(1-t, \overleftarrow{Z}_t^\lambda)|_H}_{\leq 1} |\overleftarrow{Z}_t^\lambda| \lambda \frac{1 + e^{2\lambda(t-1)}}{1 - e^{2\lambda(t-1)}} dt \right|_H^2 \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E} \exp \alpha_2^{(\lambda)} \left| \int_0^1 \frac{|\overleftarrow{Z}_t^\lambda|}{\sqrt{e^{2\lambda(1-t)} - 1}} \underbrace{\lambda (e^{2\lambda(1-t)} - 1)}_{=e^{-2\lambda(t-1)}+1} \frac{1 + e^{2\lambda(t-1)}}{1 - e^{2\lambda(t-1)}} \frac{dt}{\sqrt{e^{2\lambda(1-t)} - 1}} \right|^2 \\
&\leq \mathbb{E} \exp \alpha_2^{(\lambda)} \left| \int_0^1 \frac{|\overleftarrow{Z}_t^\lambda|}{\sqrt{e^{2\lambda(1-t)} - 1}} \lambda (e^{2\lambda(1-t)} + 1) \frac{dt}{\sqrt{e^{2\lambda(1-t)} - 1}} \right|^2.
\end{aligned}$$

Setting

$$D_\lambda := \int_0^1 \frac{dt}{\sqrt{e^{2\lambda(1-t)} - 1}} = \frac{\arctan(\sqrt{e^{2\lambda} - 1})}{\lambda} < \infty,$$

the above term can be written as

$$\mathbb{E} \exp \alpha_2^{(\lambda)} \left| \int_0^1 \frac{|\overleftarrow{Z}_t^\lambda|}{\sqrt{e^{2\lambda(1-t)} - 1}} \lambda (e^{2\lambda(1-t)} + 1) D_\lambda \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)} - 1}} \right|^2.$$

Applying Jensen's Inequality w.r.t. the probability measure $\frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)} - 1}}$ and the convex function $\exp \alpha_2^{(\lambda)} |\cdot|^2$ results in the above being bounded by the following

$$\begin{aligned}
&\mathbb{E} \int_0^1 \exp \left[\alpha_2^{(\lambda)} \left| \frac{|\overleftarrow{Z}_t^\lambda|}{\sqrt{e^{2\lambda(1-t)} - 1}} \lambda (e^{2\lambda(1-t)} + 1) D_\lambda \right|^2 \right] \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)} - 1}} \\
&= \mathbb{E} \int_0^1 \exp \left[\alpha_2^{(\lambda)} \frac{|\overleftarrow{Z}_{1-t}^\lambda|^2}{e^{2\lambda(1-t)} - 1} \lambda^2 (e^{2\lambda(1-t)} + 1)^2 D_\lambda^2 \right] \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)} - 1}}.
\end{aligned}$$

Setting $\alpha_2^{(\lambda)} := \frac{1}{4\lambda(e^{2\lambda}+1)D_\lambda^2}$ and applying Fubini's Theorem the above term can be estimated by

$$\int_0^1 \mathbb{E} \exp \left(\frac{1}{4} \frac{\lambda(e^{2\lambda(1-t)} + 1) |\overleftarrow{Z}_{1-t}^\lambda|^2}{e^{2\lambda(1-t)} - 1} \right) \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)} - 1}}. \quad (2.1.6)$$

Using [Øks10, Theorem 8.5.7] (see also Step 2 of the proof of Theorem 2.3) we have

$$\overleftarrow{Z}_{1-t}^\lambda = (2\lambda)^{-1/2} e^{-\lambda(1-t)} \bar{B}_{e^{2\lambda(1-t)} - 1},$$

where \bar{B} is another Brownian motion. Plugging this into (2.1.6) we get the following bound for (2.1.6)

$$\int_0^1 \mathbb{E} \exp \left(\frac{1}{8} \frac{\overbrace{(e^{2\lambda(1-t)} + 1)e^{-2\lambda(1-t)} \bar{B}_{e^{2\lambda(1-t)} - 1}^2}^{\leq 2}}{e^{2\lambda(1-t)} - 1} \right) \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)} - 1}}$$

$$\begin{aligned}
&\leq \int_0^1 \underbrace{\mathbb{E} \exp \left(\frac{1}{4} \frac{\bar{B}_{e^{2\lambda(1-t)}-1}^2}{e^{2\lambda(1-t)}-1} \right)}_{=\sqrt{2}} \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}} \\
&= \sqrt{2} \underbrace{\int_0^1 \frac{dt}{D_\lambda \sqrt{e^{2\lambda(1-t)}-1}}}_{=1} = \sqrt{2} =: C_2.
\end{aligned}$$

Estimate for I_3 :

Recall that

$$\mathbb{E}|I_3|_H^{2k} = \mathbb{E} \left| \int_0^1 b(s, Z_s^\lambda) dZ_s^\lambda \right|_H^{2k}. \quad (2.1.7)$$

Plugging in

$$Z_t^\lambda = -\lambda \int_0^t Z_s^\lambda ds + B_t$$

into Equation (2.1.7) results in

$$\mathbb{E}|I_3|_H^{2k} \leq 2^{2k} \mathbb{E} \left| \int_0^1 b(s, Z_s^\lambda) \lambda Z_s^\lambda ds \right|_H^{2k} + 2^{2k} \mathbb{E} \left| \int_0^1 b(s, Z_s^\lambda) dB_s \right|_H^{2k}.$$

For the first term on the right-hand side we use Jensen's Inequality and for the second term a similar calculation as for the estimate of I_1 yields that the above is smaller than

$$2^{2k} \mathbb{E} \int_0^1 \underbrace{\|b\|_\infty^{2k}}_{\leq 1} \lambda^{2k} |Z_s^\lambda|^{2k} ds + 2^{2k} 2^{5k} k!.$$

Using Fubini's Theorem we estimate this by

$$2^{2k} \lambda^{2k} \int_0^1 \mathbb{E}|Z_s^\lambda|^{2k} ds + 2^{2k} 2^{5k} k! \leq 2^{2k} \lambda^{2k} \max_{s \in [0,1]} \mathbb{E}|Z_s^\lambda|^{2k} + 2^{2k} 2^{5k} k!.$$

With the help of [Øks10, Theorem 8.5.7] (see also Step 2 of the proof of Theorem 2.3) we have

$$Z_s^\lambda = (2\lambda)^{-1/2} e^{-\lambda s} \bar{B}_{e^{2\lambda s}-1},$$

where \bar{B} is another Brownian motion. Estimating the $2k$ -moments yields

$$\begin{aligned}\mathbb{E}|Z_s^\lambda|^{2k} &= (2\lambda)^{-k} e^{-\lambda 2ks} \mathbb{E}|\bar{B}_{e^{2\lambda s}-1}|^{2k} \\ &= (2\lambda)^{-k} \underbrace{e^{-\lambda 2ks} |e^{2\lambda s} - 1|^k}_{\leq 1} 2^k \pi^{-1/2} \Gamma\left(k + \frac{1}{2}\right) \\ &\leq \lambda^{-k} \pi^{-1/2} \Gamma\left(k + \frac{1}{2}\right) \leq \lambda^{-k} k!, \quad \forall s \in [0, 1].\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\mathbb{E}|I_3|_H^{2k} &\leq 2^{2k} \lambda^{2k} \max_{s \in [0, 1]} \mathbb{E}|Z_s^\lambda|^{2k} + 2^{2k} 2^{5k} k! \\ &\leq 2^{2k} \lambda^{2k} \lambda^{-k} k! + 2^{2k} 2^{5k} k! = 2^{2k} \lambda^k k! + 2^{2k} 2^{5k} k!.\end{aligned}$$

Choosing $\alpha_3^{(\lambda)} = 2^{-6} \min(\lambda^{-1}, 2^{-2})$ we obtain

$$\mathbb{E} \exp\left(\alpha_3^{(\lambda)} |I_3|_H^2\right) = \mathbb{E} \sum_{k=0}^{\infty} \frac{|\alpha_3^{(\lambda)}|^k |I_3|_H^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{|\alpha_3^{(\lambda)}|^k \mathbb{E}|I_3|_H^{2k}}{k!} \leq \sum_{k=0}^{\infty} 2 \cdot 2^{-k} = 4 =: C_3.$$

Final estimate:

We are now ready to plug in all previous estimates to complete the proof. Setting

$$\alpha_\lambda := \frac{1}{9} \min(\alpha_1, \alpha_2^{(\lambda)}, \alpha_3^{(\lambda)})$$

we conclude

$$\begin{aligned}\mathbb{E} \exp(\alpha_\lambda |I|_H^2) &= \mathbb{E} \exp(\alpha_\lambda |I_1 + I_2 + I_3|_H^2) \leq \mathbb{E} \exp(3\alpha_\lambda |I_1|_H^2 + 3\alpha_\lambda |I_2|_H^2 + 3\alpha_\lambda |I_3|_H^2) \\ &= \mathbb{E} \exp(3\alpha_\lambda |I_1|_H^2) \exp(3\alpha_\lambda |I_2|_H^2) \exp(3\alpha_\lambda |I_3|_H^2).\end{aligned}$$

We apply the Young Inequality to split the three terms

$$\mathbb{E} \frac{\exp(3\alpha_\lambda |I_1|_H^2)^3}{3} + \mathbb{E} \frac{\exp(3\alpha_\lambda |I_2|_H^2)^3}{3} + \mathbb{E} \frac{\exp(3\alpha_\lambda |I_3|_H^2)^3}{3}$$

and using the estimates for I_1 , I_2 and I_3 results in the following bound

$$\mathbb{E} \frac{\exp(\alpha_1 |I_1|_H^2)}{3} + \mathbb{E} \frac{\exp(\alpha_2^{(\lambda)} |I_2|_H^2)}{3} + \mathbb{E} \frac{\exp(\alpha_3^{(\lambda)} |I_3|_H^2)}{3} \leq \frac{1}{3} (C_1 + C_2 + C_3) = \frac{6 + \sqrt{2}}{3} \leq 3.$$

We still need to show that the map α fulfills the claimed properties.

Simplification of α_λ :

Recall that

$$\alpha_\lambda = \frac{1}{9} \min(\alpha_1, \alpha_2^{(\lambda)}, \alpha_3^{(\lambda)}) = \frac{1}{9} \min\left(\frac{1}{256}, \frac{1}{4\lambda(e^{2\lambda} + 1)D_\lambda^2}, \frac{1}{64\lambda}\right)$$

and

$$D_\lambda = \frac{\arctan\left(\sqrt{e^{2\lambda} - 1}\right)}{\lambda}.$$

First, we want to prove that α_λ is the same as

$$\frac{1}{9} \min\left(\frac{1}{256}, \frac{1}{4\lambda(e^{2\lambda} + 1)D_\lambda^2}\right).$$

I.e. $\alpha_3^{(\lambda)}$ is always larger than α_1 or $\alpha_2^{(\lambda)}$. Note that for $\lambda \in]0, 4]$ $\alpha_3^{(\lambda)}$ is obviously larger than α_1 , hence it is enough to show that $\alpha_3^{(\lambda)} \geq \alpha_2^{(\lambda)}$ for all $\lambda > 4$. We have

$$2\lambda^2 + 2\lambda - \frac{10}{3\pi}\sqrt{16\lambda} + 2 \geq 0, \quad \forall \lambda \in \mathbb{R},$$

which implies

$$\frac{10}{3\pi}\sqrt{16\lambda} \leq 2 + 2\lambda + 2\lambda^2 \leq \sqrt{e^{2\lambda} + 1}, \quad \forall \lambda > 4.$$

Reordering and using that \arctan is an increasing function leads us to

$$\begin{aligned} \sqrt{16\lambda} &\leq \sqrt{e^{2\lambda} + 1} \frac{3\pi}{10} = \sqrt{e^{2\lambda} + 1} \arctan\left(\sqrt{1 + \frac{2}{\sqrt{5}}}\right) \\ &\leq \sqrt{e^{2\lambda} + 1} \arctan\left(\sqrt{e^2 - 1}\right) \leq \sqrt{e^{2\lambda} + 1} \arctan\left(\sqrt{e^{2\lambda} - 1}\right) \end{aligned}$$

for all $\lambda > 4$. Therefore we obtain

$$16\lambda^2 \leq (e^{2\lambda} + 1) \arctan^2\left(\sqrt{e^{2\lambda} - 1}\right),$$

which finally implies

$$\alpha_3^{(\lambda)} = \frac{1}{64\lambda} \geq \frac{\lambda}{4(e^{2\lambda} + 1) \arctan^2\left(\sqrt{e^{2\lambda} - 1}\right)} = \alpha_2^{(\lambda)}.$$

In conclusion we proved that

$$\alpha_\lambda = \frac{1}{9} \min\left(\frac{1}{256}, \frac{1}{4\lambda(e^{2\lambda} + 1)D_\lambda^2}\right).$$

Asymptotic behavior of α_λ :

Let us now analyze $\alpha_2^{(\lambda)}$. Set

$$f(\lambda) := \alpha_2^{(\lambda)} e^{2\lambda} \lambda^{-1} = \frac{e^{2\lambda}}{4\lambda^2(e^{2\lambda} + 1)D_\lambda^2} = \frac{e^{2\lambda}}{4(e^{2\lambda} + 1) \arctan^2(\sqrt{e^{2\lambda} - 1})}.$$

We obviously have

$$\frac{e^{2\lambda}}{e^{2\lambda} + 1} \xrightarrow{\lambda \rightarrow \infty} 1$$

and

$$\arctan(\sqrt{e^{2\lambda} - 1}) \xrightarrow{\lambda \rightarrow \infty} \frac{\pi}{2}.$$

Therefore,

$$f(\lambda) = \frac{e^{2\lambda}}{4(e^{2\lambda} + 1) \arctan^2(\sqrt{e^{2\lambda} - 1})} \xrightarrow{\lambda \rightarrow \infty} \frac{1}{\pi^2}$$

holds. We want to show that f is monotonically decreasing and hence the above limit is a lower bound for f . To this end we calculate the first derivative of f

$$f'(\lambda) = -\frac{e^{2\lambda} \left(e^{2\lambda} + 1 - 2 \arctan(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} \right)}{4 \arctan^3(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} (e^{2\lambda} - 1)^2}.$$

since the denominator is clearly positive, we have to show that

$$e^{2\lambda} + 1 - 2 \arctan(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} > 0, \quad \forall \lambda > 0.$$

Substituting $x := \sqrt{e^{2\lambda} - 1}$ leads to

$$x^2 + 2 > 2x \arctan(x), \quad \forall x > 0. \tag{2.1.8}$$

We prove this inequality in two steps. First note that

$$x^2 - \frac{10\pi}{12}x + 2 > 0, \quad \forall x > 0$$

holds, so that for all x with $0 < x \leq 2 + \sqrt{3}$ we have the estimate

$$x^2 + 2 > 2x \frac{5\pi}{12} = 2x \arctan(2 + \sqrt{3}) \geq 2x \arctan(x)$$

and, on the other hand, for $x \geq 2 + \sqrt{3}$ we obtain

$$x^2 + 2 \geq (2 + \sqrt{3})x + 2 > (2 + \sqrt{3})x \geq \pi x = 2x \frac{\pi}{2} > 2x \arctan(x).$$

In conclusion (2.1.8) holds, so that $f' < 0$ and therefore

$$f(\lambda) \geq \frac{1}{\pi^2}, \quad \forall \lambda > 0.$$

All together this yields

$$\alpha_\lambda e^{2\lambda} \lambda^{-1} = \frac{1}{9} \min \left(\frac{1}{256} e^{2\lambda} \lambda^{-1}, \alpha_2^{(\lambda)} e^{2\lambda} \lambda^{-1} \right) \geq \frac{1}{9} \min \left(\frac{e}{128}, \frac{1}{\pi^2} \right) = \frac{e}{1152}.$$

α_λ **is constant on** $[0, 1]$:

Claim:

$$\alpha_2^{(\lambda)} \geq \frac{1}{256}, \quad \forall \lambda \in [0, 1].$$

Let $\lambda \in [0, 1]$ and set

$$g(\lambda) := \frac{(e^{2\lambda} + 1)(e^{2\lambda} - 1)}{\lambda}.$$

g has the first derivative

$$g'(\lambda) = \frac{1 - (1 - 4\lambda)e^{4\lambda}}{\lambda^2}.$$

We want to show that $1 - (1 - 4\lambda)e^{4\lambda}$ is non-negative and thus prove that g is an non-decreasing function. To this end observe that

$$(1 - 4\lambda)e^{4\lambda}$$

is a decreasing function on $[0, \infty[$, since the derivative $-16\lambda e^{4\lambda}$ is clearly non-positive, so that

$$(1 - 4\lambda)e^{4\lambda} \leq 1$$

holds for all $\lambda \geq 0$. This leads to

$$1 - (1 - 4\lambda)e^{4\lambda} \geq 0, \quad \forall \lambda \geq 0.$$

This proves that g is non-decreasing. Using this we can easily conclude

$$\max_{\lambda \in [0, 1]} g(\lambda) \leq g(1) = (e^2 + 1)(e^2 - 1) \leq 64$$

and hence

$$g(\lambda) = \frac{(e^{2\lambda} + 1)(e^{2\lambda} - 1)}{\lambda} \leq \frac{256}{4}, \quad \forall \lambda \in [0, 1].$$

Taking the reciprocal on both sides yields

$$\alpha_2^{(\lambda)} = \frac{\lambda}{4(e^{2\lambda} + 1)(e^{2\lambda} - 1)} \geq \frac{1}{256}, \quad \forall \lambda \in [0, 1]. \quad (2.1.9)$$

Note that

$$\arctan(x) \leq x, \quad \forall x \in \mathbb{R}_+.$$

This can be proved by calculating the Taylor-polynomial up to the first order and dropping the remainder term which is always negative on \mathbb{R}_+ . Using this on our above estimate (2.1.9) we obtain

$$\frac{\lambda}{4(e^{2\lambda} + 1) \arctan^2(\sqrt{e^{2\lambda} - 1})} \geq \frac{1}{256}, \quad \forall \lambda \in [0, 1].$$

This implies that α_λ is constant on the interval $[0, 1]$.

α_λ is non-increasing:

By the previous part we can assume that $\lambda \geq 1$. We have to show that $\alpha_2^{(\lambda)}$ is non-increasing on the interval $[1, \infty[$. We do this by showing that the derivative of $\alpha_2^{(\lambda)}$

$$\begin{aligned} \left(\alpha_2^{(\lambda)}\right)' &= - \frac{\overbrace{2\lambda}^{=: p_1} - \overbrace{\arctan(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1}}^{=: n_1}}{4 \arctan^3(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} (e^{2\lambda} + 1)^2} \\ &\quad - \frac{\overbrace{2\lambda e^{2\lambda}}^{=: p_2} - \overbrace{\arctan(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} e^{2\lambda}}^{=: n_2} + \overbrace{2\lambda \arctan(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} e^{2\lambda}}^{=: p_3}}{4 \arctan^3(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} (e^{2\lambda} + 1)^2}. \end{aligned}$$

is non-positive. So, to simplify notation we have to show that

$$p_1 - n_1 + p_2 - n_2 + p_3 \geq 0, \quad \forall \lambda \geq 1 \tag{2.1.10}$$

holds. Note that for $\lambda \geq 1$

$$p_3 - n_1 - n_2 \geq \arctan(\sqrt{e^{2\lambda} - 1}) \sqrt{e^{2\lambda} - 1} e^{2\lambda} (2\lambda - 2) \geq 0,$$

so that (2.1.10) holds, which finishes the proof that $\alpha_2^{(\lambda)}$ is non-increasing on $[1, \infty[$. Together with the previous established result that α is constant on $[0, 1]$ this completes the proof that α_λ is non-increasing on \mathbb{R}_+ . □

Lemma 2.2

Let $(Z_t^A)_{t \in [0, \infty[}$ be a H -valued Ornstein–Uhlenbeck process with drift term A as explained in

the introduction, i.e.

$$\begin{cases} dZ_t^A = -AZ_t^A dt + dB_t, \\ Z_0^A = 0. \end{cases}$$

Let $(\lambda_n)_{n \in \mathbb{N}}$ be the eigenvalues of A . Let $C \in \mathbb{R}$ and the map α be as in Proposition 2.1. Then for all Borel measurable functions $b: [0, 1] \times H \rightarrow H$, which are in the second component twice continuously differentiable with

$$\|b\|_\infty := \sup_{t \in [0, 1], x \in \mathbb{R}} |b(t, x)|_H \leq 1$$

we have

$$\mathbb{E} \exp \left(\alpha_{\lambda_i} \left| \int_0^1 \partial_{x_i} b(t, Z_t) dt \right|_H^2 \right) \leq C \leq 3 \quad \forall i \in \mathbb{N},$$

where $\partial_{x_i} b$ denotes the derivative of b w.r.t. the i -th component of the second parameter x .

Proof

Let

$$Z_t^A = (Z_t^{A, (n)})_{n \in \mathbb{N}} \in \ell^2 \cong H$$

be the components of $(Z_t^A)_{t \in [0, \infty[}$ and $(\lambda_n)_{n \in \mathbb{N}}$ be the eigenvalues of A w.r.t. the basis $(e_n)_{n \in \mathbb{N}}$. Note that every component $Z^{A, (n)}$ is a one-dimensional Ornstein–Uhlenbeck process with drift term $\lambda_n > 0$ driven by the one-dimensional Brownian motion $B^{(n)}$. Define $\tilde{B}^{(n)}$ by

$$\tilde{B}_t^{(n)} := \int_0^{\gamma^{(n)}(t)} \sqrt{c^{(n)}(s)} dB_s^{(n)}, \quad \forall t \in [0, 1],$$

where

$$\gamma^{(n)}(t) := (2\lambda_n)^{-1} \ln(t + 1) \quad \text{and} \quad c^{(n)}(t) := (2\lambda_n) e^{2\lambda_n t}.$$

Observe that

$$(\gamma^{(n)}(t))' = \frac{1}{c^{(n)}(\gamma^{(n)}(t))}$$

and, hence, by [Øks10, Theorem 8.5.7] $(\tilde{B}_t^{(n)})_{t \in [0, \infty[}$ is a Brownian motion for every $n \in \mathbb{N}$.

Therefore, the components of Z^A can be written as time-transformed Brownian motions

$$Z_t^{A, (n)} = (2\lambda_n)^{-1/2} e^{-\lambda_n t} \tilde{B}_{e^{2\lambda_n t} - 1}^{(n)},$$

since

$$\begin{aligned} (2\lambda_n)^{-1/2} \tilde{B}_{e^{2\lambda_n t}-1}^{(n)} &= (2\lambda_n)^{-1/2} \int_0^t (2\lambda_n)^{1/2} e^{\lambda_n s} dB_s^{(n)} \\ &= \int_0^t e^{\lambda_n s} dB_s^{(n)} = Z_t^{A,(n)} e^{\lambda_n t}. \end{aligned}$$

Let us define the mapping

$$\begin{aligned} \varphi_A: \mathcal{C}([0, \infty[, H) &\longrightarrow \mathcal{C}([0, \infty[, H) \\ f = (f^{(n)})_{n \in \mathbb{N}} &\longmapsto \left(t \longmapsto \left((2\lambda_n)^{-1/2} e^{-\lambda_n t} f^{(n)}(e^{2\lambda_n t} - 1) \right)_{n \in \mathbb{N}} \right). \end{aligned}$$

φ_A is bijective and we have used that $\mathcal{C}([0, \infty[, \mathbb{R}^{\mathbb{N}}) \cong \mathcal{C}([0, \infty[, \mathbb{R})^{\mathbb{N}}$ as topological spaces. By definition of the product topology φ_A is continuous if and only if $\pi_n \circ \varphi_A$ is continuous for every $n \in \mathbb{N}$.

$$\begin{array}{ccc} \mathcal{C}([0, \infty[, H) & \xrightarrow{\varphi_A} & \mathcal{C}([0, \infty[, H) \\ & \searrow \varphi_A^{(n)} := \pi_n \circ \varphi_A & \downarrow \pi_n \\ & & \mathcal{C}([0, \infty[, \mathbb{R}) \end{array}$$

Here, π_n denotes the projection to the n -th component. The above mapping φ_A is continuous and, therefore, measurable w.r.t. the Borel sigma-algebra. Using this transformation, the Ornstein–Uhlenbeck measure \mathbb{P}_A , as defined in the introduction, can be written as

$$\mathbb{P}_A[F] = Z^A(\mathbb{P})[F] = (\varphi_A \circ \tilde{B})(\mathbb{P})[F] = \varphi_A(\mathcal{W})[F], \quad \forall F \in \mathcal{B}(\mathcal{C}([0, \infty[, H)),$$

because of

$$Z_t^A = \varphi_A \circ \tilde{B}_t.$$

Hence, we have

$$\mathbb{P}_A = \varphi_A(\mathcal{W}) = \varphi_A \left(\bigotimes_{n \in \mathbb{N}} \mathcal{W}^{(n)} \right) = \bigotimes_{n \in \mathbb{N}} \varphi_A^{(n)}(\mathcal{W}^{(n)}), \quad (2.2.1)$$

where $\mathcal{W}^{(n)}$ is the projection of \mathcal{W} to the n -th coordinate and the last equality follows from

$$\int_F d\varphi_A \left(\bigotimes_{n \in \mathbb{N}} \mathcal{W}^{(n)} \right) = \prod_{n \in \mathbb{N}} \int_{\pi_n(\varphi_A^{-1}(F))} d\mathcal{W}^{(n)} = \prod_{n \in \mathbb{N}} \int_{(\varphi_A^{(n)})^{-1}(\pi_n(F))} d\mathcal{W}^{(n)} = \left(\bigotimes_{n \in \mathbb{N}} \varphi_A^{(n)}(\mathcal{W}^{(n)}) \right) [F].$$

Starting from the left-hand side of the assertion we have

$$\mathbb{E} \exp \alpha_{\lambda_i} \left| \int_0^1 \partial_{x_i} b(t, Z_t^A) dt \right|_H^2.$$

Using Equation (2.2.1) we can write this as

$$\int_{\mathcal{C}([0, \infty[, \mathbb{R})^{\mathbb{N}})} \exp \alpha_{\lambda_i} \left| \int_0^1 \partial_{x_i} b \left(t, ((\varphi_A^{(n)} \circ f_n)(t))_{n \in \mathbb{N}} \right) dt \right|_H^2 d \bigotimes_{n \in \mathbb{N}} \mathcal{W}^{(n)}(f_n),$$

where $(f_n)_{n \in \mathbb{N}}$ are the components of f . Using Fubini's Theorem we can perform the i -th integral first and obtain

$$\int_{\mathcal{C}([0, \infty[, \mathbb{R})^{\mathbb{N} \setminus \{i\}})} \int_{\mathcal{C}([0, \infty[, \mathbb{R})} \exp \alpha_{\lambda_i} \left| \int_0^1 \partial_{x_i} b \left(t, ((\varphi_A^{(n)} \circ f_n)(t))_{n \in \mathbb{N}} \right) dt \right|_H^2 d\mathcal{W}^{(i)}(f_i) d \bigotimes_{\substack{n \in \mathbb{N} \\ n \neq i}} \mathcal{W}^{(n)}(f_n).$$

Since $\varphi_A^{(i)} \circ f_i$ is under $\mathcal{W}^{(i)}$ distributed as $Z^{A, (i)}$ under \mathbb{P} . By Proposition 2.1 the inner integral is smaller than C , so that the entire expression is smaller than

$$\int_{\mathcal{C}([0, \infty[, \mathbb{R})^{\mathbb{N} \setminus \{i\}})} C d \bigotimes_{\substack{n \in \mathbb{N} \\ n \neq i}} \mathcal{W}^{(n)}(f_n) = C,$$

where in the last step we used that $\mathcal{W}^{(n)}$ are probability measures.

□

Theorem 2.3

Let $\ell \in]0, 1]$ and $(Z_t^{\ell A})_{t \in [0, \infty[}$ be an H -valued Ornstein–Uhlenbeck process with drift term ℓA , i.e.

$$\begin{cases} dZ_t^{\ell A} = -\ell A Z_t^{\ell A} dt + dB_t, \\ Z_0^{\ell A} = 0. \end{cases}$$

There exists an absolute constant $C \in \mathbb{R}$ (independent of A and ℓ) such that for all Borel measurable functions $b: [0, 1] \times H \longrightarrow H$ with

$$\|b\|_{\infty} := \sup_{t \in [0, 1], x \in H} |b(t, x)|_H \leq 1$$

and

$$\|b\|_{\infty, A} := \sup_{t \in [0, 1], x \in H} \left(\sum_{n \in \mathbb{N}} \lambda_n e^{2\lambda_n} b_n(t, x)^2 \right)^{1/2} \leq 1.$$

This means

$$(\lambda_n^{1/2} e^{\lambda_n} b_n(t, x))_{n \in \mathbb{N}} \in \ell^2 \cong H, \quad \forall (t, x) \in [0, 1] \times H,$$

where b_n is the n -th component of b . The following inequality

$$\mathbb{E} \exp \frac{\beta_A}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A}) dt \right|_H^2 \leq C \leq 3,$$

where

$$\beta_A := \frac{1}{4} \Lambda^{-2} \inf_{n \in \mathbb{N}} \alpha_{\lambda_n} e^{2\lambda_n} \lambda_n^{-1} > 0$$

holds uniformly for all bounded, measurable functions $h: [0, 1] \rightarrow H$ with

$$\|h\|_\infty := \sup_{t \in [0, 1]} |h(t)|_H \in]0, \infty[$$

and

$$\sum_{n \in \mathbb{N}} |h_n(t)|^2 \lambda_n^2 < \infty, \quad \forall t \in [0, 1].$$

Recall that Λ is defined in Equation (1.0.1) and the map α is from Proposition 2.1.

Proof

Step 1: The case for twice continuously differentiable b .

Let $Z^{\ell A}$ be an H -valued Ornstein–Uhlenbeck process, $b: [0, 1] \times H \rightarrow H$ a bounded, Borel measurable function which is twice continuously differentiable in the second component with $\|b\|_\infty \leq 1$, $\|b\|_{\infty, A} \leq 1$ and $h: [0, 1] \rightarrow H$ a bounded, measurable function with $\|h\|_\infty \neq 0$. Let α and C be as in Proposition 2.1. Recall that Λ is defined as

$$\Lambda = \sum_{n \in \mathbb{N}} \lambda_n^{-1} < \infty.$$

Note that by Proposition 2.1 $\beta_A > 0$. By the Fundamental Theorem of Calculus we obtain

$$\begin{aligned} & \mathbb{E} \exp \frac{4\beta_A}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A}) dt \right|_H^2 \\ &= \mathbb{E} \exp \frac{4\beta_A}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^{\ell A} + \theta h(t)) \Big|_{\theta=0}^{\theta=1} dt \right|_H^2 \end{aligned}$$

$$= \mathbb{E} \exp \frac{4\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \int_0^1 b'(t, Z_t^{\ell A} + \theta h(t)) h(t) \, d\theta dt \right|_H^2,$$

where b' denotes the Fréchet derivative of b w.r.t. x . Using Fubini's Theorem we can switch the order of integration, so that the above equals

$$\begin{aligned} & \mathbb{E} \exp 4\beta_A \left| \int_0^1 \int_0^1 b'(t, Z_t^{\ell A} + \theta h(t)) \frac{h(t)}{\|h\|_\infty} \, dt d\theta \right|_H^2 \\ &= \mathbb{E} \exp 4\beta_A \left| \int_0^1 \int_0^1 \sum_{i \in \mathbb{N}} \underbrace{b'(t, Z_t^{\ell A} + \theta h(t)) e_i}_{=\partial_{x_i} b(t, Z_t^{\ell A} + \theta h(t))} \frac{h_i(t)}{\|h\|_\infty} \, dt d\theta \right|_H^2 \\ &= \mathbb{E} \exp 4\beta_A \left| \int_0^1 \int_0^1 \sum_{i \in \mathbb{N}} \frac{h_i(t)}{\|h\|_\infty} \sum_{j \in \mathbb{N}} \partial_{x_i} b_j(t, Z_t^{\ell A} + \theta h(t)) e_j \, dt d\theta \right|_H^2 \\ &= \mathbb{E} \exp 4\beta_A \left| \int_0^1 \int_0^1 \sum_{i \in \mathbb{N}} \lambda_i^{-1/2} \partial_{x_i} \underbrace{\frac{h_i(t)}{\|h\|_\infty} \lambda_i^{1/2} \sum_{j \in \mathbb{N}} b_j(t, Z_t^{\ell A} + \theta h(t)) e_j}_{=e^{-\lambda_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A})} \, dt d\theta \right|_H^2, \end{aligned} \quad (2.3.1)$$

where

$$\tilde{b}_{h,\theta,i}(t, x) := e^{\lambda_i} \frac{h_i(t)}{\|h\|_\infty} \lambda_i^{1/2} \sum_{j \in \mathbb{N}} b_j(t, x + \theta h(t)) e_j.$$

Note that $\|\tilde{b}_{h,\theta,i}\|_\infty \leq 1$ because for all $(t, x) \in [0, 1] \times H$ we have

$$\begin{aligned} |\tilde{b}_{h,\theta,i}(t, x)|_H &= \underbrace{\frac{|h_i(t)|}{\|h\|_\infty}}_{\leq 1} \lambda_i^{1/2} e^{\lambda_i} \left| \sum_{j \in \mathbb{N}} b_j(t, x + \theta h(t)) e_j \right|_H \\ &\leq \lambda_i^{1/2} e^{\lambda_i} \left(\sum_{j \in \mathbb{N}} \lambda_j^{-1} e^{-2\lambda_j} \lambda_j e^{2\lambda_j} b_j(t, x + \theta h(t))^2 \right)^{1/2} \\ &\leq \underbrace{\lambda_i^{1/2} e^{\lambda_i} \sup_{j \in \mathbb{N}} \lambda_j^{-1/2} e^{-\lambda_j}}_{\leq 1} \underbrace{\left(\sum_{j \in \mathbb{N}} \lambda_j e^{2\lambda_j} b_j(t, x + \theta h(t))^2 \right)^{1/2}}_{\leq 1} \leq 1. \end{aligned}$$

Using Jensen’s Inequality and again Fubini’s Theorem the Expression (2.3.1) is bounded from above by

$$\int_0^1 \mathbb{E} \exp 4\beta_A \left| \sum_{i \in \mathbb{N}} \lambda_i^{-1/2} \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta.$$

Applying Hölder Inequality we can split the sum and estimate this from above by

$$\begin{aligned} & \int_0^1 \mathbb{E} \exp 4\beta_A \underbrace{\sum_{i \in \mathbb{N}} \lambda_i^{-1}}_{=\Lambda} \sum_{i \in \mathbb{N}} \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta \\ &= \int_0^1 \mathbb{E} \exp 4\beta_A \Lambda \sum_{i \in \mathbb{N}} \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta \\ &= \int_0^1 \mathbb{E} \prod_{i \in \mathbb{N}} \exp 4\beta_A \Lambda \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta. \end{aligned}$$

Young’s Inequality with $p_i := \lambda_i \Lambda$ leads us to the upper bound

$$\begin{aligned} & \int_0^1 \mathbb{E} \sum_{i \in \mathbb{N}} \frac{1}{p_i} \exp 4\beta_A \Lambda p_i \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta. \\ &= \int_0^1 \sum_{i \in \mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp 4\beta_A \Lambda^2 \lambda_i \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta. \end{aligned} \tag{2.3.2}$$

Recall that

$$\beta_A = \frac{1}{4} \Lambda^{-2} \inf_{n \in \mathbb{N}} \alpha_{\lambda_n} e^{2\lambda_n} \lambda_n^{-1},$$

hence, we can estimate (2.3.2) from above by

$$\begin{aligned} & \int_0^1 \sum_{i \in \mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp \alpha_{\lambda_i} e^{2\lambda_i} \left| \int_0^1 e^{-\lambda_i} \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta. \\ &= \int_0^1 \sum_{i \in \mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp \alpha_{\lambda_i} \left| \int_0^1 \partial_{x_i} \tilde{b}_{h,\theta,i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta. \end{aligned}$$

Since $\ell \in]0, 1]$ and α is non-increasing by Proposition 2.1 the above is smaller than

$$\int_0^1 \sum_{i \in \mathbb{N}} \frac{1}{p_i} \mathbb{E} \exp \alpha_{\ell \lambda_i} \left| \int_0^1 \partial_{x_i} \tilde{b}_{h, \theta, i}(t, Z_t^{\ell A}) dt \right|_H^2 d\theta.$$

Applying Lemma 2.2 for every $\theta \in [0, 1]$ and $i \in \mathbb{N}$ results in the estimate

$$\int_0^1 \underbrace{\sum_{i \in \mathbb{N}} \frac{1}{p_i}}_{=1} C d\theta = C.$$

Step 2: The general case: Non-smooth b .

Let $b: [0, 1] \times H \longrightarrow H$ be a bounded, Borel measurable function with $\|b\|_\infty \leq 1$, $\|b\|_{\infty, A} \leq 1$ and $h: [0, 1] \longrightarrow H$ a bounded, Borel measurable function with $0 \neq \|h\|_\infty < \infty$ and

$$\sum_{n \in \mathbb{N}} |h_n(t)|^2 \lambda_n^2 < \infty \quad \forall t \in [0, 1].$$

Let β_A and C be the constants from Step 1. Set $\varepsilon := \exp \frac{-64\beta_A}{\|h\|_\infty^2}$ as well as

$$\mu_0 := dt \otimes Z_t^{\ell A}(\mathbb{P}),$$

$$\mu_h := dt \otimes (Z_t^{\ell A} + h(t))(\mathbb{P}).$$

Note that the measure $Z_t^{\ell A}(\mathbb{P})$ is equivalent to the invariant measure $N(0, \frac{1}{2\ell} A^{-1})$ due to [DZ92, Theorem 11.13] and analogously $(Z_t^{\ell A} + h(t))(\mathbb{P})$ to $N(h(t), \frac{1}{2\ell} A^{-1})$. Furthermore, $h(t)$ is in the domain of A for every $t \in [0, 1]$ because of

$$\sum_{n \in \mathbb{N}} \langle h(t), e_n \rangle^2 \lambda_n^2 \leq \sum_{n \in \mathbb{N}} |h_n(t)|^2 \lambda_n^2 < \infty.$$

We set

$$g(t) := 2\ell A h(t).$$

Observe that $g(t) \in H$ for every $t \in [0, 1]$ because of

$$|g(t)|_H^2 = 4\ell^2 \sum_{n \in \mathbb{N}} \lambda_n^2 |h_n(t)|^2 < \infty.$$

Hence, [Bog98, Corollary 2.4.3] is applicable i.e. $N(0, \frac{1}{2\ell} A^{-1})$ and $(Z_t^{\ell A} + h(t))(\mathbb{P})$ are equivalent measures. By the Radon–Nikodym Theorem there exist a density ρ so that

$$\frac{d\mu_h}{d\mu_0} = \rho.$$

Furthermore, there exists $\delta > 0$ such that

$$\int_A \rho \, d\mu_0(t, x) \leq \frac{\varepsilon}{2}, \quad (2.3.3)$$

for all measurable sets $A \subseteq [0, 1] \times H$ with $\mu_0[A] \leq \delta$. Set

$$\bar{\delta} := \min \left(\delta, \frac{\varepsilon}{2} \right). \quad (2.3.4)$$

By Lusin’s Theorem (see [Tao11, Theorem 1.3.28]) there exist a closed set $K \subseteq [0, 1] \times H$ with $\mu_0[K] \geq 1 - \bar{\delta}$ such that the restriction

$$b|_K : K \longrightarrow H, \quad (t, x) \longmapsto b(t, x)$$

is continuous. Note that

$$(\mu_0 + \mu_h)[K^c] = \underbrace{\mu_0[K^c]}_{\leq \bar{\delta} \leq \frac{\varepsilon}{2}} + \mu_h[K^c] \leq \frac{\varepsilon}{2} + \underbrace{\int_{K^c} \rho \, d\mu_0(t, x)}_{\leq \frac{\varepsilon}{2} \text{ by (2.3.4) and (2.3.3)}} \leq \varepsilon. \quad (2.3.5)$$

Applying Dugundji’s Extension Theorem (see [Dug51, Theorem 4.1]) to the function $b|_K$ guarantees that there exists a continuous function $\bar{b} : [0, 1] \times H \longrightarrow H$ with $\|\bar{b}\|_\infty \leq \|b\|_\infty$ and $\|\bar{b}\|_{\infty, A} \leq \|b\|_{\infty, A}$ which coincides with b on K . Starting from the left-hand side of the assertion we have

$$\mathbb{E} \exp \frac{\beta_A}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A}) \, dt \right|_H^2.$$

Adding and subtracting \bar{b} and using that $b - \bar{b} = 0$ on K yields that the above equals

$$\begin{aligned} \mathbb{E} \exp \frac{\beta_A}{\|h\|_\infty^2} & \left| \int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) \underbrace{[b(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A} + h(t))]}_{\in [-2, 2]} \right. \\ & \quad \left. - \mathbb{1}_{K^c}(t, Z_t^{\ell A}) \underbrace{[b(t, Z_t^{\ell A}) - \bar{b}(t, Z_t^{\ell A})]}_{\in [-2, 2]} \right|_H^2 \\ & + \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) \, dt \Big|_H^2. \end{aligned}$$

Applying the fact that $(a + b)^2 \leq 2a^2 + 2b^2$ we estimate from above by

$$\begin{aligned}
& \mathbb{E} \exp \left(\frac{8\beta_A}{\|h\|_\infty^2} \left(\int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) dt \right)^2 \right. \\
& \quad \left. + \frac{2\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) dt \right|_H^2 \right) \\
&= \mathbb{E} \exp \left(\frac{8\beta_A}{\|h\|_\infty^2} \left(\int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) dt \right)^2 \right) \\
& \quad \cdot \exp \left(\frac{2\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) dt \right|_H^2 \right)
\end{aligned}$$

and using Young's Inequality this is bounded by

$$\begin{aligned}
& \underbrace{\frac{1}{2} \mathbb{E} \exp \left(\frac{16\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) dt \right|_H^2 \right)}_{=: A_1} \\
& + \underbrace{\frac{1}{2} \mathbb{E} \exp \left(\frac{4\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}(t, Z_t^{\ell A} + h(t)) - \bar{b}(t, Z_t^{\ell A}) dt \right|_H^2 \right)}_{=: A_2}.
\end{aligned}$$

Let us estimate A_1 first

$$\begin{aligned}
A_1 &= 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{16\beta_A}{\|h\|_\infty^2} \right)^k \mathbb{E} \left| \int_0^1 \mathbb{1}_{K^c}(t, Z_t^{\ell A} + h(t)) + \mathbb{1}_{K^c}(t, Z_t^{\ell A}) dt \right|_H^{2k} \\
&\leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{16\beta_A}{\|h\|_\infty^2} \right)^k 2^{2k} \underbrace{(\mu_h[K^c] + \mu_0[K^c])}_{\leq \varepsilon \text{ by (2.3.5)}} \leq 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{64\beta_A}{\|h\|_\infty^2} \right)^k \varepsilon \\
&\leq 1 + \exp \left(\frac{64\beta_A}{\|h\|_\infty^2} \right) \varepsilon = 1 + 1 = 2.
\end{aligned}$$

This concludes the estimate for A_1 . Let us now estimate A_2 . Since \bar{b} is continuous there exists a sequence $\bar{b}^{(m)} : [0, 1] \times H \rightarrow H$ of functions with $\|\bar{b}^{(m)}\|_\infty \leq 1$ and $\|\bar{b}^{(m)}\|_{\infty, A} \leq 1$ which are smooth in the second component (i.e. twice continuously differentiable) such that $\bar{b}^{(m)}$ converges to \bar{b} everywhere, i.e.

$$\bar{b}^{(m)}(t, x) \xrightarrow{m \rightarrow \infty} \bar{b}(t, x), \quad \forall t \in [0, 1], \forall x \in H.$$

Using the above considerations A_2 equals

$$\mathbb{E} \exp \frac{4\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \lim_{m \rightarrow \infty} \bar{b}^{(m)}(t, Z_t^{\ell A} + h(t)) - \bar{b}^{(m)}(t, Z_t^{\ell A}) dt \right|_H^2,$$

which in turn can be bounded using Fatou's Lemma by

$$\liminf_{m \rightarrow \infty} \mathbb{E} \exp \frac{4\beta_A}{\|h\|_\infty^2} \left| \int_0^1 \bar{b}^{(m)}(t, Z_t^{\ell A} + h(t)) - \bar{b}^{(m)}(t, Z_t^{\ell A}) dt \right|_H^2. \quad (2.3.6)$$

Applying Step 1 with b replaced by $\bar{b}^{(m)}$ yields that (2.3.6) and henceforth A_2 is bounded by C , so that in conclusion we have

$$\mathbb{E} \exp \frac{\beta_A}{\|h\|_\infty^2} \left| \int_0^1 b(t, Z_t^{\ell A} + h(t)) - b(t, Z_t^{\ell A}) dt \right|_H^2 \leq \frac{1}{2}A_1 + \frac{1}{2}A_2 \leq 1 + \frac{C}{2} \leq 3,$$

which completes the proof. □

3 A concentration of measure result

For this section let us define

$$Z^A(t, x) := Z_t^A + e^{-tA}x, \quad \forall x \in H, t \in [0, \infty[$$

then for every $x \in H$, $Z(\cdot, x)$ is an Ornstein–Uhlenbeck process starting in x . Furthermore, we define the image measure

$$\mathbb{P}_x := \mathbb{P} \circ Z^A(\cdot, x)^{-1}, \quad \forall x \in H$$

and the projections

$$\pi_t(f) := f(t), \quad \forall f \in \mathcal{C}([0, \infty[, H), t \in [0, \infty[,$$

which come with their canonical filtration

$$\bar{\mathcal{G}}_t := \sigma(\pi_s | s \leq t),$$

so that $(\mathcal{C}([0, \infty[, H), (\mathbb{P}_x)_{x \in H}, (\pi_t)_{t \in [0, \infty[}, (\bar{\mathcal{G}}_t)_{t \in [0, \infty[})$ is a universal Markov process (see [LR15, Proposition 4.3.5] and [Bau96, Section 42] or [Jac05, Section 3.4] for the definition of a universal Markov process). Additionally, we set

$$\mathcal{G}_t := \{Z^{-1}(B) | B \in \bar{\mathcal{G}}_t\}$$

as the initial sigma-algebra, so that Z becomes $\mathcal{G}_t/\bar{\mathcal{G}}_t$ -measurable.

Corollary 3.1

There exists $\beta_A > 0$ (depending only on the drift term A of the Ornstein–Uhlenbeck process Z^A) such that for all $0 \leq r < u \leq 1$ and for any bounded Borel measurable function $b: [r, u] \times H \rightarrow H$ with $\|b\|_\infty \leq 1$, $\|b\|_{\infty, A} \leq 1$, any bounded Borel measurable functions $h_1, h_2: [r, u] \rightarrow H$ with

$$\sum_{n \in \mathbb{N}} |h_{1,n}(t)|^2 \lambda_n^2 + \sum_{n \in \mathbb{N}} |h_{2,n}(t)|^2 \lambda_n^2 < \infty, \quad \forall t \in [0, 1].$$

for any $\eta \geq 0$ the inequality

$$\mathbb{P} \left[\left| \int_r^u b(s, Z_s^A + h_1(s)) - b(s, Z_s^A + h_2(s)) \, ds \right|_H > \eta \ell^{1/2} \|h_1 - h_2\|_\infty \middle| \mathcal{G}_r \right] \leq 3e^{-\beta_A \eta^2}$$

holds, where $\ell := u - r$.

Proof

Let r, u, ℓ, b, h_1 and h_2 be as in the assertion. Note that the assertion is trivial if $h_1 = h_2$, hence w.l.o.g. we assume $\|h_1 - h_2\|_\infty \neq 0$. We define the stochastic processes $\tilde{Z}_t^{\ell A} := \ell^{-1/2} Z_{\ell t}^A$ and $\tilde{B}_t := \ell^{-1/2} B_{\ell t}$. Note that \tilde{B} is again a Brownian motion w.r.t. the normal, right-continuous filtration $(\tilde{\mathcal{F}}_t^\ell)_{t \in [0, \infty[} := (\mathcal{F}_{\ell t})_{t \in [0, \infty[}$. Additionally, we have

$$\begin{aligned} \tilde{Z}_t^{\ell A} &= \ell^{-1/2} Z_{\ell t}^A = \ell^{-1/2} \int_0^{\ell t} e^{(\ell t - s)A} \, dB_s \\ &= \int_0^{\ell t} e^{\ell(t - \frac{s}{\ell})A} \ell^{-1/2} \, dB_{\frac{s}{\ell}} = \int_0^t e^{(t - s')\ell A} \, d\tilde{B}_{s'}. \end{aligned}$$

Hence, $\tilde{Z}^{\ell A}$ is an Ornstein–Uhlenbeck process with drift term ℓA .

For the reader's convenience we add the integration variable as a superscript to the respective measure which we integrate against, hence the left-hand side of the claim reads

$$\mathbb{P}^{(\mathrm{d}\omega)} \left[\left| \int_r^u b(s, Z_s^A(\omega) + h_1(s)) - b(s, Z_s^A(\omega) + h_2(s)) \, \mathrm{d}s \right|_H > \eta \ell^{1/2} \|h_1 - h_2\|_\infty \middle| \mathcal{G}_r \right].$$

Fix an $\omega' \in \Omega$. Using the transformation $s' := \ell^{-1}(s - r)$ this equals

$$\mathbb{P}^{(\mathrm{d}\omega)} \left[\left| \ell \int_0^1 b(\ell s' + r, Z_{\ell s' + r}^A(\omega) + h_1(\ell s' + r)) - b(\ell s' + r, Z_{\ell s' + r}^A(\omega) + h_2(\ell s' + r)) \, \mathrm{d}s' \right|_H > \eta \ell^{1/2} \|h_1 - h_2\|_\infty \middle| \mathcal{G}_r \right] (\omega').$$

Recall the definitions of π_t and $\bar{\mathcal{G}}_t$ at the beginning of this section. Since \mathcal{G}_t is the initial sigma-algebra of $\bar{\mathcal{G}}_t$ w.r.t. Z^A we have

$$\mathbb{E} [\pi_t \circ Z^A | \mathcal{G}_r] (\omega') = \mathbb{E}_0 [\pi_t | \bar{\mathcal{G}}_r] (Z^A(\omega')),$$

where \mathbb{E}_0 denotes the expectation w.r.t. the measure \mathbb{P}_0 . Applying this to the above situation we obtain that the left-hand side of the assertion reads

$$\mathbb{P}_0^{(\mathrm{d}\omega)} \left[\left| \int_0^1 b(\ell s + r, \pi_{\ell s + r}(\omega) + h_1(\ell s + r)) - b(\ell s + r, \pi_{\ell s + r}(\omega) + h_2(\ell s + r)) \, \mathrm{d}s \right|_H > \eta \ell^{-1/2} \|h_1 - h_2\|_\infty \middle| \bar{\mathcal{G}}_r \right] (Z^A(\omega')),$$

Applying the universal Markov property (see [Bau96, Equation (42.18)] or [Jac05, Equation (3.108)]) we have

$$\begin{aligned} &= \mathbb{P}_{\pi_r(Z^A(\omega'))}^{(\mathrm{d}\omega)} \left[\left| \int_0^1 b(\ell s + r, \pi_{\ell s}(\omega) + h_1(\ell s + r)) - b(\ell s + r, \pi_{\ell s}(\omega) + h_2(\ell s + r)) \, \mathrm{d}s \right|_H > \eta \ell^{-1/2} \|h_1 - h_2\|_\infty \right]. \end{aligned} \quad (3.1.1)$$

We define

$$\tilde{b}(t, x) := b(\ell t + r, \ell^{1/2} x),$$

$$\tilde{h}_1(t) := \ell^{-1/2} h_1(\ell t + r),$$

$$\tilde{h}_2(t) := \ell^{-1/2} h_2(\ell t + r),$$

so that the Expression (3.1.1) simplifies to

$$\mathbb{P}_{\pi_r(Z^A(\omega'))}^{(\mathrm{d}\omega)} \left[\left| \int_0^1 \tilde{b}(s, \ell^{-1/2} \pi_{\ell s} + \tilde{h}_1(s)) - \tilde{b}(s, \ell^{-1/2} \pi_{\ell s} + \tilde{h}_2(s)) \, \mathrm{d}s \right|_H > \eta \left\| \tilde{h}_1 - \tilde{h}_2 \right\|_\infty \right].$$

Note that $\tilde{b}, \tilde{h}_1, \tilde{h}_2$ are all bounded Borel measurable functions and $\|\tilde{b}\|_\infty = \|b\|_\infty \leq 1$ as well as $\|\tilde{b}\|_{\infty, A} = \|b\|_{\infty, A} \leq 1$ hold. Plugging in the definition of \mathbb{P}_x the above reads

$$\begin{aligned} & (\mathbb{P} \circ Z^A(\cdot, Z_r^A(\omega'))^{-1})^{(\mathrm{d}\omega)} \left[\left| \int_0^1 \tilde{b}(s, \ell^{-1/2} \pi_{\ell s}(\omega) + \tilde{h}_1(s)) \right. \right. \\ & \quad \left. \left. - \tilde{b}(s, \ell^{-1/2} \pi_{\ell s}(\omega) + \tilde{h}_2(s)) \, \mathrm{d}s \right|_H > \eta \left\| \tilde{h}_1 - \tilde{h}_2 \right\|_\infty \right] \\ &= \mathbb{P} \left[\left| \int_0^1 \tilde{b}_{\omega', \tilde{h}_2}(s, \underbrace{\ell^{-1/2} Z^A(\ell s, Z_r^A(\omega')) - \ell^{-1/2} e^{-\ell s A} Z_r^A(\omega')}_{=\ell^{-1/2} Z_{\ell s}^A = \tilde{Z}_s^{\ell A}} + \tilde{h}_1(s) - \tilde{h}_2(s)) \right. \right. \\ & \quad \left. \left. - \tilde{b}_{\omega', \tilde{h}_2}(s, \underbrace{\ell^{-1/2} Z^A(\ell s, Z_r^A(\omega')) - \ell^{-1/2} e^{-\ell s A} Z_r^A(\omega')}_{=\ell^{-1/2} Z_{\ell s}^A = \tilde{Z}_s^{\ell A}}) \, \mathrm{d}s \right|_H > \eta \left\| \tilde{h}_1 - \tilde{h}_2 \right\|_\infty \right], \end{aligned}$$

where $\tilde{b}_{\omega', \tilde{h}_2}(t, x) := \tilde{b}(t, x + \ell^{-1/2} e^{-\ell t A} Z_r^A(\omega') + \tilde{h}_2(t))$. Recall that $\tilde{Z}^{\ell A}$ is an Ornstein–Uhlenbeck process which starts in 0. By Theorem 2.3 there exist constants β_A (depending on the drift term A , but independent of ℓ since $\ell \in]0, 1]$) and C such that the conclusion of Theorem 2.3 holds for every Ornstein–Uhlenbeck process $\tilde{Z}^{\ell A}$ with the same constants β_A and C . Since $\exp(\beta_A |\cdot|^2)$ is increasing on \mathbb{R}_+ the above equals

$$\mathbb{P} \left[\exp \left(\frac{\beta_A}{\left\| \tilde{h}_1 - \tilde{h}_2 \right\|_\infty^2} \left| \int_0^1 \tilde{b}_{\omega', \tilde{h}_2}(s, \tilde{Z}_s^{\ell A} + \tilde{h}_1(s) - \tilde{h}_2(s)) - \tilde{b}_{\omega', \tilde{h}_2}(s, \tilde{Z}_s^{\ell A}) \, \mathrm{d}s \right|_H^2 \right) > \exp(\beta_A \eta^2) \right]$$

and by Chebyshev's Inequality this can be estimated from above by

$$e^{-\beta_A \eta^2} \mathbb{E} \exp \left(\frac{\beta_A}{\|\tilde{h}_1 - \tilde{h}_2\|_\infty^2} \left| \int_0^1 \tilde{b}_{\omega', \tilde{h}_2}(s, \tilde{Z}_s^{\ell A} + \tilde{h}_1(s) - \tilde{h}_2(s)) - \tilde{b}_{\omega', \tilde{h}_2}(s, \tilde{Z}_s^{\ell A}) \, ds \right|_H^2 \right).$$

Since $\|\tilde{b}_{\omega', \tilde{h}_2}\|_\infty = \|\tilde{b}\|_\infty \leq 1$ as well as $\|\tilde{b}_{\omega', \tilde{h}_2}\|_{\infty, A} = \|\tilde{b}\|_{\infty, A} \leq 1$ holds the conclusion of Theorem 2.3 implies that the above expression is smaller than

$$C e^{-\beta_A \eta^2} \leq 3 e^{-\beta_A \eta^2}.$$

□

Corollary 3.2

For all $0 \leq r < u \leq 1$ and for every bounded Borel measurable function $b: [r, u] \times H \rightarrow H$ with $\|b\|_\infty \leq 1$, $\|b\|_{\infty, A} \leq 1$ and for all bounded \mathcal{G}_r -measurable random variables $x, y: \Omega \rightarrow H$ with

$$\sum_{n \in \mathbb{N}} |x_n|^2 \lambda_n^2 + \sum_{n \in \mathbb{N}} |y_n|^2 \lambda_n^2 < \infty.$$

We have

$$\mathbb{E} \left[\left| \int_r^u b(s, Z_s^A + x) - b(s, Z_s^A + y) \, ds \right|_H^p \middle| \mathcal{G}_r \right] \leq 3 \beta_A^{p/2} p^{p/2} \ell^{p/2} |x - y|_H^p,$$

where $\ell := u - r$, $p \in \mathbb{N}$ and $\beta_A > 0$ is the constant from Corollary 3.1.

Proof

Let $0 \leq r < u \leq 1$ and b, p, ℓ as in the assertion.

Step 1: Deterministic x, y

Let $x, y \in H$ be non-random with $x \neq y$. We set

$$S := \beta_A^{1/2} \ell^{-1/2} |x - y|_H^{-1} \left| \int_r^u b(s, Z_s^A + x) - b(s, Z_s^A + y) \, ds \right|_H$$

and calculate

$$\mathbb{E}[S^p | \mathcal{G}_r] = \mathbb{E} \left[\int_0^\infty \mathbb{1}_{\{S > \eta\}} p \eta^{p-1} d\eta \middle| \mathcal{G}_r \right].$$

Notice that the above is valid since S is a non-negative random variable. Using Fubini's Theorem the above equals

$$\int_0^\infty p \eta^{p-1} \mathbb{P}[S > \eta | \mathcal{G}_r] d\eta.$$

Plugging in the definition of S the above line reads

$$\int_0^\infty p \eta^{p-1} \mathbb{P} \left[\left| \int_r^u b(s, Z_s^A + x) - b(s, Z_s^A + y) ds \right|_H > \beta_A^{-1/2} \eta \ell^{1/2} |x - y|_H \middle| \mathcal{G}_r \right] d\eta.$$

We estimate the probability inside the integral by invoking Corollary 3.1, hence the above expression is smaller than

$$3 \int_0^\infty p \eta^{p-1} e^{-\eta^2} d\eta = \frac{3p}{2} \Gamma\left(\frac{p}{2}\right).$$

Using Stirling's formula this is bounded from above by

$$\frac{3}{2} p \underbrace{\sqrt{\frac{4\pi}{p}} 2^{-p/2} e^{-p/2} e^{\frac{1}{6p}}}_{\leq \sqrt{2\pi} e^{-1/2} e^{\frac{1}{6}}} p^{p/2} \leq 3p^{p/2},$$

which proves that $\mathbb{E}[S^p | \mathcal{G}_r] \leq 3\beta_A^{p/2} p^{p/2}$, concluding the assertion in the case that x and y are deterministic.

Step 2: Random x, y

Let $x, y: \Omega \rightarrow H$ be \mathcal{G}_r measurable random variables of the form

$$x = \sum_{i=1}^n \mathbb{1}_{A_i} x_i, \quad y = \sum_{i=1}^n \mathbb{1}_{A_i} y_i,$$

where $x_i, y_i \in H$ and $(A_i)_{1 \leq i \leq n}$ are pairwise disjoint sets in \mathcal{G}_r . Notice that due to the disjointness we have

$$b\left(t, Z_t^A + \sum_{i=1}^n \mathbb{1}_{A_i} x_i\right) - b\left(t, Z_t^A + \sum_{i=1}^n \mathbb{1}_{A_i} y_i\right) = \sum_{i=1}^n \mathbb{1}_{A_i} [b(t, Z_t^A + x_i) - b(t, Z_t^A + y_i)].$$

Let p be a positive integer. Starting from the left-hand side of the assertion and using the above identity yields

$$\begin{aligned} & \mathbb{E} \left[\left| \int_r^u b(t, Z_t^A + x) - b(t, Z_t^A + y) dt \right|_H^p \middle| \mathcal{G}_r \right] \\ &= \sum_{i=1}^n \mathbb{E} \left[\mathbb{1}_{A_i} \left| \int_r^u b(t, Z_t^A + x_i) - b(t, Z_t^A + y_i) dt \right|_H^p \middle| \mathcal{G}_r \right]. \end{aligned}$$

Since $A_i \in \mathcal{G}_r$ this can be expressed as

$$\sum_{i=1}^n \mathbb{1}_{A_i} \mathbb{E} \left[\left| \int_r^u b(t, Z_t^A + x_i) - b(t, Z_t^A + y_i) dt \right|_H^p \middle| \mathcal{G}_r \right]$$

and by invoking Step 1 this is bounded from above by

$$3\beta_A^{p/2} p^{p/2} \ell^{p/2} \sum_{i=1}^n \mathbb{1}_{A_i} |x_i - y_i|_H^p = 3\beta_A^{p/2} p^{p/2} \ell^{p/2} |x - y|_H^p.$$

In conclusion we obtained the result for step functions x, y . The result for general \mathcal{G}_r measurable random variables x, y now follows by approximation via step functions and taking limits. \square

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